Int. J. Solids Structures, 1971, Vol. 7, pp. 1411 to 1425. Pergamon Press. Printed in Great Britain

MINIMUM WEIGHT DESIGN OF CONTINUOUS BEAMS

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Abstract—If frictionless hinges are placed at all the points in the beam where the bending moment is zero, it is shown that the number and positions of these hinges are such as to render the elastic or plastic minimum weight beam statically determinate. This fact can be used to obtain particular designs. In the plastic design a collapse mechanism exists such that there is no change in slope at these hinges. In the elastic design the deformed beam will in general have a change in slope at these points, although under special circumstances the change is zero. Both the elastic and plastic designs are shown to be fully stressed.

INTRODUCTION

THE study of the minimum weight design of structures is of interest for two reasons. The primary one is of course the hope that the minimum weight design obtained can actually be used in a practical design. Even if this is not the case, for example if the structure has to be too idealized in order to facilitate the analysis, an absolute minimum weight is established which can be used to measure the efficiency of the practical design. Secondly, general design rules may evolve which can help to guide the designer when a complete minimum weight design is not carried out. An example of this is the rule which states that a minimum weight truss subject to fixed loads must be statically determinate [1]. It is in this spirit that the present paper is submitted.

The paper presented is concerned with the elastic and plastic minimum weight design of continuous beams with a continuously varying cross section subject to given constant loads. The design of this type of beam has been the subject of several papers. The majority of these papers consider the plastic design of beams where the weight per unit length is proportional to the full plastic moment of the cross section, as is the case for a rectangular sandwich beam with fixed core size and identical face sheets of variable thickness. For such beams a sufficient condition for minimum weight is that there can be found a plastic collapse mechanism with a curvature rate of constant absolute value such that the bending moment and curvature have the same sign. This theorem was first proved by Heyman [2]. In the present paper an analogous necessary condition for beams where the weight per unit length is a general monotonically increasing function of the fully plastic moment is presented.

The main part of the paper is divided into three parts. In the first part, the governing equations for the problem are set down. In the second part the minimum weight design of perfectly plastic beams are investigated. It is shown that if weightless hinges are placed at all the zero moment points in the beam, the position and number of these hinges are such that the beam is statically determinate. In the third part the relationship between

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the elastic and plastic designs are investigated. It is shown that if these hinges are frictionless and are allowed to rotate as the beam is loaded, the elastically designed minimum weight beam is fully stressed and also statically determinate in the above sense. This fact simplifies the elastic or plastic design of more complicated beam–load systems considerably particularly for the nonlinear cases. Two examples demonstrating the use of this method in simple designs are given.

FORMULATION OF THE MINIMUM WEIGHT PROBLEM

The structures to be considered are continuous beams of the type shown in Fig. 1. The end conditions of the beam, the number and the length of the spans as well as the type of loading is arbitrary. The magnitude of the applied load is constant.

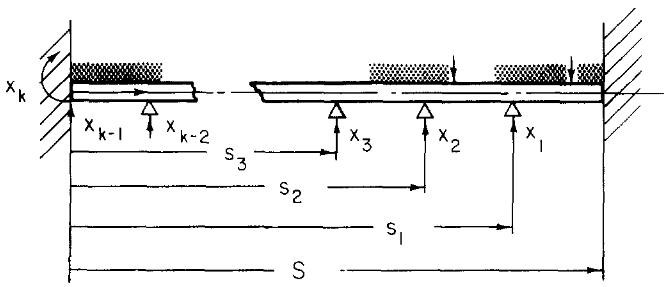

FIG. 1. Typical beam to be designed for minimum weight.

The bending moment at each point in the beam satisfies the equilibrium conditions. For a k times statically indeterminate beam, the bending moment M can be written as

$$M(s, X_i) = \sum_{i=1}^{k} b_i(s)X_i + g(s) \tag{1}$$

where s is a coordinate measured from one end of the beam as shown in Fig. 1. The X_i's are the redundant support forces. $g(s)$ is the moment resulting from the applied load. A set of $b_i(s)$ functions can in general be chosen in a number of ways. If the beam is simply supported at the left end ($s = 0$) in Fig. 1, it is convenient to write the functions as

$$\begin{aligned} b_n(s) &= 0 \quad \text{for } 0 \leq s \leq s_n \\ &= (s - s_n) \quad \text{for } s_n < s \leq S \end{aligned} \tag{2}$$

where $n = 1, \ldots k$.

As shown in Fig. 1, s_n is the coordinate of the nth redundant support counted from the right hand side. If the left end ($s = 0$) of the beam is built in, it is convenient to write $b_i(s)$ as given by (2) when $n = 1, \ldots, k-1$, and as

$$b_k(s) = 1 \quad \text{for } 0 \leq s \leq S. \tag{3}$$

If the beam is built in at the right end ($s = S$) each interval $(s_n - s_{n+1})$ corresponds to a span. If the beam is simply supported at the right end ($s = S$), this is true for all the intervals except $(S - s_1)$ which corresponds to two spans.

For a safe design the bending moment must satisfy

$$|M(s, X_i)| \leq M_0(s), \quad \text{where } M_0 \geq 0 \tag{4}$$

at each point in the beam. $M_0(s)$ is a critical moment which is a function of the material and geometric properties at a point in the beam. The influence of shear forces and axial forces (if there are any) on the strength of the beam is therefore neglected. The choice of a critical moment will be discussed further below.

The weight w per unit length of the beam is taken to be

$$w = h(M_0), \quad \text{where } h(0) = 0 \tag{5}$$

$h(M_0)$ is a strictly monotonically increasing function differentiable everywhere except at zero. The order at which $h(M_0)$ tends to zero is α where

$$0 < \alpha \leq 1. \tag{6}$$

The above function includes as a special case the linear function most commonly used in minimum weight design problems which corresponds to a rectangular sandwich beam with fixed core height and width and identical face sheets of variable thickness. Included are also geometrically similar cross sections and rectangular cross sections with constant width where the functions take the forms respectively

$$w \doteq M_0^{\frac{2}{3}} \quad \text{and} \quad w \doteq M_0^{\frac{1}{2}}. \tag{7}$$

For a more complete discussion of the weight of beams see [5].

The total weight W of the beam is

$$W = \int_0^S h(M_0(s))\, \mathrm{d}s. \tag{8}$$

The compatibility conditions for the beam can be obtained from the theorem of virtual work

$$\int_0^S k(s) b_i(s)\, \mathrm{d}s + \sum_{j=1}^{m} \theta_j b_i(s_j) = 0, \qquad i = 1, \ldots k \tag{9}$$

where $k(s)$ is the curvature of the beam and θ_j the rotation of any hinges in the beam. When the compatibility conditions are written in this form the singularities of the curvature are contained in the hinge rotations, so

$$k(s_j) = 0, \qquad j = 1, \ldots m. \tag{10}$$

The case of complete sections of the beam vanishing is not included in (9). This case is however discussed further below.

For the elastic beam

$$k = \frac{M}{EI} \tag{11}$$

where EI is the elastic stiffness of the beam.

For a rigid-perfectly plastic beam where no unloading of the material is taking place

$$\begin{aligned} k &\geq 0 \quad \text{when } M = M_0 \\ k &\leq 0 \quad \text{when } M = -M_0 \\ k &= 0 \quad \text{when } -M_0 < M < M_0 \end{aligned} \tag{12}$$

where M_0 now is the fully plastic moment of the cross-section.

The plastic minimum weight design is obtained by minimizing (8) subject to the constraints (1) and (4). M_0 is in this case the fully plastic moment.

The elastic minimum weight design is obtained by minimizing (8) subject to the constraints (1), (4) and (9). In this case the critical moment M_0 is equal to a moment which the designer considers unsafe to exceed, for example the moment which will first cause yielding or some fraction of it.

PROPERTIES OF THE PLASTIC MINIMUM WEIGHT BEAM

For the subsequent discussion it is convenient to eliminate the critical moment M_0 from the formulation of the problem. Since the weight function $h(M_0)$ is monotonically increasing, it follows from (8) that a reduction is M_0 at any point will reduce the weight of the beam. In the minimum weight design M_0 must therefore be as small as the constraints (1) and (4) will permit.

Since (1) does not involve M_0 it follows that

$$M_0(s) = |M(X_i, s)| \quad \text{for } 0 \leq s \leq S. \tag{13}$$

A necessary condition for a plastic minimum weight design is therefore that the moment everywhere is equal to its critical value, that is, the beam must be fully stressed.

The minimum weight design is now reduced to minimizing

$$W(X_i) = \int_0^S h[|M(X_i, s)|] \, \mathrm{d}s = \int_0^S f(M(X_i, s)) \, \mathrm{d}s \tag{14}$$

where $M(X_i, s)$ is given by (1). The values of X_i are unrestricted. The function $f(M)$ is differentiable everywhere except at $M = 0$. Since the weight $W(X_i)$ is always nonnegative and since it is finite at $X_i = 0$ and becomes very large when X_i becomes very large, it is clear that the minimum weight design exists and that it occurs for finite values of X_i.

If $M = 0$ only at a finite number of isolated points in the minimum weight beam, Megarets and Hodge [4] have shown that the weight function $W(X_i)$ in equation (14) is continuously differentiable with respect to all X_i. A necessary condition for minimum weight is therefore

$$\frac{\partial W}{\partial X_i} = \frac{\partial}{\partial X_i} \int_0^S f(M(X_i, s)) \, \mathrm{d}s = 0, \qquad i = 1, \ldots k. \tag{15}$$

For the present analysis it is useful to restate (15) in the following form: a necessary condition for optimum design is that there exists a function $\pi(s)$ which satisfies

$$\int_0^S \pi(s) b_i(s) \, \mathrm{d}s = 0, \qquad i = 1, \ldots k \tag{16}$$

where

$$\begin{aligned} \pi(s) &= f'(M) \quad \text{when } M \neq 0 \\ \pi(s) &= 0 \qquad\quad \text{when } M = 0. \end{aligned} \tag{17}$$

Since $h(M_0)$ is strictly monotonically increasing

$$\begin{aligned} \pi(s) &> 0 \quad \text{when } M > 0 \\ \pi(s) &< 0 \quad \text{when } M < 0 \\ \pi(s) &= 0 \quad \text{when } M = 0. \end{aligned} \tag{18}$$

If $M = 0$ over a finite length in the minimum weight beam, the parts of the beam separated by these portions become in effect isolated from each other. Equations (16) and (17) are still necessary conditions for a minimum in the separate parts when the functions $b_i(s)$ are redefined in the same way as before for these portions.

If the weight function (14) is convex the conditions (16) and (17) are necessary and sufficient. This is the case for example when the weight is a linear function of the critical moment. The conditions (17) then simply reduce to

$$\begin{aligned} \pi(s) &= 1 \qquad \text{when } M > 0 \\ \pi(s) &= -1 \quad \text{when } M < 0 \\ \pi(s) &= 0 \qquad \text{when } M = 0. \end{aligned} \tag{19}$$

In the Appendix it is shown that the following four properties of the minimum weight beam follow directly from the conditions (16) and (17).

1. In the part of the beam $s_n < s < S$ the number of zeros in the bending moment is at least n.
2. There can never be two adjacent intervals (an interval is defined as $s_{n+1} < s < s_n$) without any zeros in the bending moment.
3. For each interval without any zeros in the bending moment there must be one interval with at least two zeros.
4. The total number of zeros in the bending moment is at least equal to the degree of redundance of the beam.

Each point in the beam with a zero moment can be thought of as a hinge since the beam has zero thickness at these points but is still able to transmit the shear force. Alternatively, weightless and frictionless hinges can be thought of as inserted at these points. In this sense the above properties of the beam insures that the beam is statically determinate. This is clearly true for each subsection of a beam separated by finite length zero weight sections, as well as for the whole beam including any such zero weight sections. A necessary condition for minimum weight is therefore that the beam is statically determinate in the above sense.

The plastic minimum weight design can therefore be approached by inserting hinges in a sufficient number of places and positions to render the beam statically determinate. This will in general result in a number of possible designs. The minimum weight of each of these designs can be obtained by minimizing with respect to the position of the zero moment points. The final design is then the one among these with the lowest weight. Except for the linear case, the final design will in general not be unique.

In view of equations (12), (13) and (18) it is clear that it is always possible to find a curvature for a rigid-perfectly plastic beam such that

$$k(s) \doteq \pi(s). \tag{20}$$

Using equation (16) the compatibility conditions (9) can now be separated as

$$\int_0^S k(s)b_i(s)\,\mathrm{d}s = 0 \tag{21}$$

$$\sum_{j=1}^{m} \theta_j b_i(s_j) = 0 \tag{22}$$

where $m \geq k$ and $i = 1, \ldots k$.

When $m = k$ equation (22) shows that all the hinge rotations are zero. When $m > k$ the beam has in effect become a mechanism with $(m-k)$ degrees of freedom. This is reflected in the equations (22) which allows the same degree of freedom in the choice of hinge rotations. However, zero hinge rotation at all the hinges is still a possible displacement scheme.

For the linear case, equations (19)–(21) are the same as the sufficient condition for a minimum given by Heyman [2]. In view of the above derivation, it is clear that this criterion is also a necessary one. The above argument shows that the displacement curves of constant $|k|$ must be joined with continuous slope at the points where $M = 0$, as was also pointed out by Heyman [2]. The equations (17) and (20) and (21) are the analogous necessary conditions for the general case. The procedure used by Heyman [2] to solve actual problems was first to find displacement curves with $|k| = 1$ satisfying compatibility and then to look for equilibrium moment distributions satisfying (19). However as Heyman points out using this method except for the simplest cases is quite difficult if at all possible. In the nonlinear case this method would of course no longer be possible to use since the magnitude of the curvature now depends on the bending moment which is not known *a priori.*

PROPERTIES OF THE ELASTIC MINIMUM WEIGHT BEAM

It will now be investigated under what circumstances a beam which satisfies the necessary conditions (16) and (17) for plastic minimum weight and which is therefore statically determinate in the above sense also satisfies the elastic compatibility conditions (9).

Two distinct cases occur. In the first case the elastic curvature k given by equation (11) is proportional to the function π defined in equation (17). When this is the case the separated compatibility conditions equations (21) and (22) also holds for the elastic case and the argument about hinge rotations is unchanged. The proportionality of the two functions will occur in a variety of circumstances of practical interest. In particular, if the weight functions (5) is of the simple form

$$w \doteq M_0^{\alpha}, \quad \text{where } 0 < \alpha \leq 1. \tag{23}$$

The two functions are proportional if

$$\frac{I}{M_0^{2-\alpha}} = \text{const.} \tag{24}$$

If furthermore I and M_0 are functions of a single design parameter t such that

$$I(t, s) = t^g(s), \qquad M_0(t, s) = t^p(s) \tag{25}$$

equation (24) reduces to

$$g-(2-\alpha)p = 0. \tag{26}$$

This is satisfied for example by a sandwich section with constant core dimensions ($g = 1$, $p = 1$, $\alpha = 1$), for a rectangular section with constant width ($g = 3$, $p = 2$, $\alpha = \frac{1}{2}$) and for geometrically similar section ($g = 4$, $p = 3$, $\alpha = \frac{2}{3}$). Therefore, in this case the necessary conditions (16) and (17) also applies to an elastic design with a displacement curve with a continuous slope.

In the second and general case, k and π are not proportional and the compatibility conditions (9) cannot be divided into two separate parts as above. These equations will now determine the necessary hinge rotations to insure elastic compatibility. When $m = k$ the rotations are given uniquely by (9). When $m > k$ there is again for the same reason as above, $(m-k)$ degrees of freedom in the choice of the rotations.

In general therefore, if hinge rotations at the zero moment points are admissible the elastic and plastic minimum weight design can be approached in essentially the same way. The design method will only differ in the choice of the critical moment M_0. Both designs are in this case fully stressed. The elastic design obtained in this way will be lighter than or have the same weight as one obtained demanding a displacement curve with a continuous slope.

Masur [3] has shown that a necessary condition for an elastically designed structure to have a maximum stiffness for a given weight (in the sense of minimum work done by the applied loads) is that the total strain energy density in the design fibers of the structure is constant. For the case with a single design parameter t this means that

$$\frac{M^2(\mathrm{d}I/\mathrm{d}t)}{I(t)} = \text{const.} \tag{27}$$

For a fully stressed design where equation (25) is satisfied this reduces to

$$2p-g = 1. \tag{28}$$

This is satisfied by the sandwich beam and the rectangular beam with constants width, but not by the geometrically similar beam. For the class of beams which satisfy (27) the possibility exists that the stiffest beam is also the strongest. The result is however not certain since both conditions are only necessary. However, as was shown by Masur [3] when the weight function and the moment of inertia are convex the above condition is also sufficient. For the sandwich beam, which satisfies this, the stiffest beam is therefore also the strongest.

EXAMPLES

Some simple examples may illustrate the ideas introduced.

Consider the four span beam subject to a single concentrated load P shown in Fig. 2. Take the weight per unit length to be a linear function of the critical moment.

From the properties of the elastic or plastic minimum weight beam demonstrated it follows that there must be at least one zero in the first interval, that is between C and E

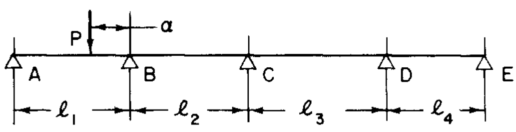

FIG. 2. Four span beam.

in Fig. 2. If the zero is in the span DE the moment must be zero everywhere in that span, and if it is zero in CD it must be zero in both spans. In each case the weight of the span DE is zero. This span can therefore be removed. There must now be at least one zero between B and D. By the same argument as before, this means that the span CD can be removed. The resulting structure is one times statically indeterminate, and must therefore have at least one zero in the bending moment. Three possibilities exist. The bending moments for these three cases are shown in Figs. 3(a)–(c). The weights W can be determined directly for cases a and b

$$W_a = \frac{P\alpha}{2}(l_2+\alpha) \tag{29}$$

$$W_b = \frac{P\alpha}{2}(l_1-\alpha) \tag{30}$$

where l_1 and l_2 are the length of the spans AB and BC and α gives the position of the load.

For case c the weight depends on the position β of the zero moment point. The value of β resulting in a minimum weight can be found by minimizing the weight with respect to

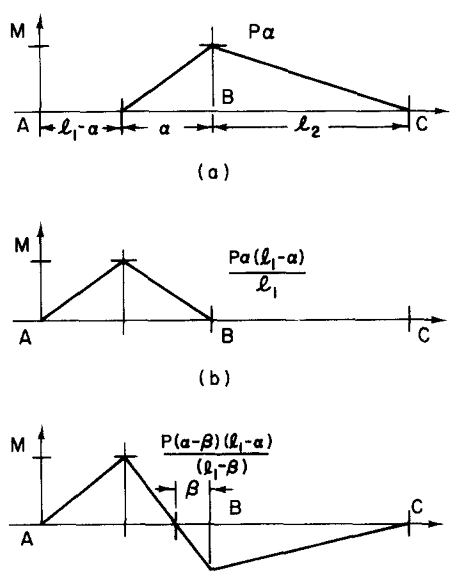

FIG. 3. Possible bending moments in minimum weight beam.

the position β of the zero point. In this way

$$W_c = \frac{P(l_1-\alpha)[(\alpha-\beta)(l_1-\beta)+\beta(\beta+l_2)]}{2(l_1-\beta)} \tag{31}$$

where

$$\frac{\beta}{l_1} = 1-\left[\frac{1}{2}\left(1+\frac{l_2}{l_1}\right)\right]^{\frac{1}{2}} \tag{32}$$

To have a physical meaning the value of β must be $0 \leq \beta \leq \alpha$. This means that

$$\frac{l_2}{l_1} \geq \frac{\alpha}{l_1} \geq 1-\left[\frac{1}{2}\left(1+\frac{l_2}{l_1}\right)\right]^{\frac{1}{2}}. \tag{33}$$

The minimum weight design is now that one of the three cases with the smallest weight. The weight depends on the position of the load and the relative length of the spans l_1 and l_2. The result is shown graphically in Fig. 4.

Since the weight function is linear, this is also the stiffest beam for this weight. The elastic and plastic displacement curves also have continuous slopes through the zero moment points.

As a second example consider the built-in beam shown in Fig. 5. This is the same example considered by Masur [3] using a different approach to determine the elastic optimum strength and stiffness. The results obtained are the same.

Let the beam have a rectangular cross section with a constant width a and height h to be determined. The beam is to be designed elastically with a maximum allowable stress σ_{cr}. In this case the weight per unit length is

$$W = \rho\left(\frac{6a}{\sigma_{cr}}\right)^{\frac{1}{2}} M^{\frac{1}{2}} \tag{34}$$

where ρ is the density.

From the properties of the minimum weight beam it follows that the bending moment must be zero in at least two places in the beam. From the linear form of the moment it is clear that there must be just two zero points, and that they must be symmetrically placed

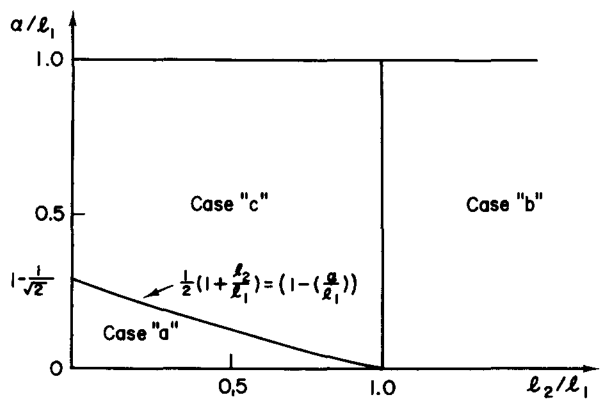

FIG. 4. Minimum weight regimes.

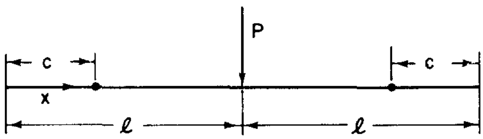

FIG. 5. One span built in beam.

about the middle. The position of the zero points can be found by minimizing the weight with respect to the position c of the zero moment points.

$$W = \frac{4\rho}{3}\left(\frac{6a}{\sigma_{cr}}\right)^{\frac{1}{2}}\left(\frac{P}{2}\right)^{\frac{1}{2}}\{c^{\frac{3}{2}}+(l-c)^{\frac{3}{2}}\} \tag{35}$$

which has a minimum when

$$c = \frac{l}{2}. \tag{36}$$

The height of the minimum weight beam follows from the bending moment

$$h = \left(\frac{3P}{a\sigma_{cr}}\right)^{\frac{1}{2}}\left(\left|\frac{l}{2}-x\right|\right)^{\frac{1}{2}}. \tag{37}$$

This is shown in Fig. 6.

In this example equations (16) and (17) are satisfied for only one position of the zero moment points. (There are no local minimum weight configurations with at least two zero moment points.) The solution is therefore also the stiffest solution since equation (28) is satisfied as well.

SUMMARY AND CONCLUSIONS

It has been shown that if frictionless hinges are placed at all the points in the beam where the bending moment is zero, the number and position of these hinges are such as to

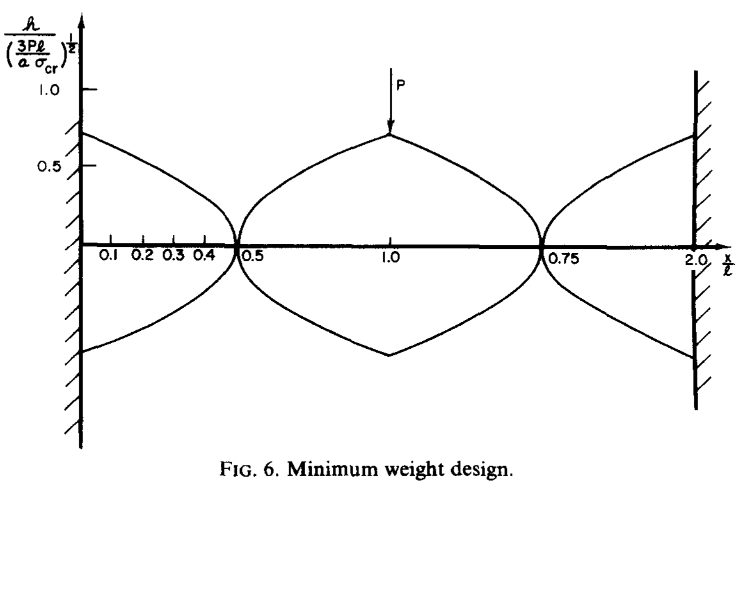

FIG. 6. Minimum weight design.

render the elastic or plastic minimum weight beam statically determinate. In this sense both the elastic and plastic minimum weight beams are fully stressed.

In the plastic beam a collapse mechanism exists such that there is no change in slope at these hinges. In the elastic design, however, there is in general a change in slope at these points, although under special circumstances the change is zero.

The minimum weight design of an originally k times statically indeterminate beam can therefore be approached by making the moment zero in k places and then minimizing the weight with respect to the position of the zero moments.

A necessary condition for minimum weight of beams where the weight per unit length is a general monotonically increasing function of the critical moment is given. This condition is analogous to the sufficient condition given earlier by Heyman [2].

Since the above properties of the minimum weight beam is proved for arbitrary applied loads it is clear that they are therefore also true for beams loaded by their own weight in addition to given fixed loads. Furthermore, since the final design with the hinges is statically determinate it is also the optimum design in the presence of thermal effects and support settlements.

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APPENDIX

Properties of the minimum weight beam

The properties of the minimum weight beam listed in the main section will be shown to follow directly from the seven theorems given and proved below.

1. The bending moment must be zero at least once in the interval $s_1 < s < S$.
2. If at $s = s_n$

$$\operatorname{sgn}\left(\int_{s_n}^{S} \pi(s)\,\mathrm{d}s\right) = \operatorname{sgn}(\pi(s_n)) \tag{38}$$

the bending moment must be zero at least once in the interval $s_{n+1} < s < s_n$.

3. If the bending moment is zero once in the interval $s_1 < s < S$ equation (38) is satisfied at $s = s_1$.

4. If equation (38) is satisfied at $s = s_n$ and the bending moment is zero once in the interval $s_{n+1} < s < s_n$ it is also satisfied at $s = s_{n+1}$.

5. If the bending moment is not zero anywhere in the interval $s_{n+1} < s < s_n$, equation (38) is satisfied at $s = s_{n+1}$.

6. If the bending moment is not zero anywhere in the interval $s_{n+1} < s < s_n$, equation (38) is not satisfied at $s = s_n$.

7. For a beam built in at both ends and for a beam with at least two spans and built in at one end there must be at least one interval with at least two zeros in the bending moment.

Consider theorem 1

Taking account of equation (2), equation (16) can be written for $i = 1$.

$$\int_0^S \pi(s) b_1(s)\, \mathrm{d}s = \int_{s_1}^S \pi(s)(s - s_1)\, \mathrm{d}s = 0. \tag{39}$$

Since $(s - s_1) \geq 0$ in the interval $s_1 \leq s \leq S$, the above integral can only be zero if $\pi(s)$ changes sign at least once in this interval. From (18) it can be seen that a change in sign of $\pi(s)$ corresponds to a point at zero bending moment. The theorem 1 is therefore true.

Consider theorem 2

Taking account of the equation (2) equation (16) can be written for $i = n+1$.

$$\int_0^S \pi(s) b_{n+1}(s)\, \mathrm{d}s = \int_{s_{n+1}}^S \pi(s)(s - s_{n+1})\, \mathrm{d}s = 0$$

or

$$\int_{s_{n+1}}^{s_n} \pi(s)(s - s_{n+1})\, \mathrm{d}s + (s_n - s_{n+1}) \int_{s_n}^S \pi(s)\, \mathrm{d}s = 0 \tag{40}$$

where $(s_n - s_{n+1}) > 0$ and $(s - s_{n+1}) \geq 0$ in the interval $s_{n+1} \leq s \leq s_n$. If equation (38) holds at $s = s_n$ the sign of $\pi(s_n)$ and the last integral in (40) have the same sign. The sum of the integrals in (40) can therefore only be zero if $\pi(s)$ changes sign at least once in the interval $s_{n+1} \leq s \leq s_n$. The theorem 2 is therefore true.

Consider theorem 3

Call the first integral in equation (38) G. Let $\pi(s)$ change sign at $s = \alpha$ where $s_1 < \alpha < S$. Introduce the new coordinate

$$s^* = \frac{s - s_1}{\alpha - s_1}$$

The integral G and equation (40) can now be written

$$\frac{G}{(\alpha - s_1)} = \int_0^1 \pi^*(s^*)\, \mathrm{d}s^* + \int_1^{S^*} \pi^*(s^*)\, \mathrm{d}s^* \tag{41}$$

$$\int_0^1 \pi^*(s^*) s^*\, \mathrm{d}s^* + \int_1^{S^*} \pi^*(s^*) s^*\, \mathrm{d}s^* = 0 \tag{42}$$

where $\pi^*(s^*) = \pi(s)$, $S^* = S - s_1/\alpha - s_1$, $S^* > 1$ and $(\alpha - s_1) > 0$. Since $\pi(s)$ only changes sign once, the sign of $\pi(s)$ is constant in each of the above integrals. It follows that

$$\begin{aligned} \left| \int_0^1 \pi^*(s^*)\, \mathrm{d}s^* \right| &> \left| \int_0^1 \pi^*(s^*) s^*\, \mathrm{d}s \right| \\ \left| \int_1^{S^*} \pi^*(s^*)\, \mathrm{d}s^* \right| &< \left| \int_1^{S^*} \pi^*(s^*) s^*\, \mathrm{d}s \right| \end{aligned} \tag{43}$$

By comparing equations (41) and (42) in view of (43) it is clear that G always takes the sign of $\pi(s_1)$. The theorem 3 is therefore true.

Consider theorem 4

Let $\pi(s)$ change sign at $s = \alpha$ where $s_{n+1} < \alpha < s_n$. Introduce the new coordinate

$$s^* = \frac{s - s_{n+1}}{\alpha - s_{n+1}}.$$

The integral G and equation (40) can now be written

$$\frac{G}{(\alpha - s_{n+1})} = \int_0^1 \pi^*(s^*)\,ds^* + \int_1^{s_n^*} \pi^*(s^*)\,ds^* + \int_{s_n^*}^{S^*} \pi^*(s^*)\,ds^* \tag{44}$$

and

$$\int_0^1 \pi^*(s^*)s^*\,ds^* + \int_1^{s_n^*} \pi^*(s^*)s^*\,ds^* + \frac{(s_n - s_{n+1})}{(\alpha - s_{n+1})}\int_{s_n^*}^{S^*} \pi^*(s^*)\,ds^* = 0 \tag{45}$$

where

$$s_n^* = \frac{s_n - s_{n+1}}{a - s_{n+1}} \quad \text{and } s_n^* > 1.$$

Since $\pi(s)$ changes sign only once in the interval $s_{n+1} \le s \le s_n$ the sign of $\pi(s)$ is constant in each of the first two integrals in (44) and (45). Furthermore if equation (38) holds at $s = s_n$ the sign of the last two integrals must be the same. It follows that

$$\left|\int_0^1 \pi^*(s^*)\,ds^*\right| > \left|\int_0^1 \pi^*(s^*)s^*\,ds^*\right| \tag{46}$$

$$\left|\int_1^{s_n^*} \pi^*(s^*)\,ds^* + \int_{s_n^*}^{S^*} \pi^*(s^*)\,ds^*\right| < \left|\int_1^{s_n^*} \pi^*(s^*)s^*\,ds^* + \frac{(s_n - s_{n+1})}{(\alpha - s_{n+1})}\int_{s_n^*}^{S^*} \pi^*(s^*)\,ds^*\right| \tag{47}$$

By comparing equations (44) and (45) in view of (40) and (47) it is clear that G always takes the sign of $\pi(s_{n+1})$. The theorem 4 is therefore true.

Consider theorem 5

Let $\pi(s)$ have no sign changes in the interval $s_{n+1} \le s \le s_n$. Introduce the new coordinate

$$s^* = \frac{s - s_{n+1}}{s_n - s_{n+1}}.$$

The integral G and equation (34) can now be written as

$$G = \int_0^1 \pi^*(s^*)\,ds^* + \int_1^{S^*} \pi^*(s^*)\,ds^* \tag{48}$$

and

$$\int_0^1 \pi^*(s^*)s^*\,ds^* + \int_1^{S^*} \pi^*(s^*)\,ds^* = 0 \tag{49}$$

where

$$S^* = \frac{S - s_{n+1}}{s_n - s_{n+1}} \quad \text{and } S^* > 1.$$

Substituting from the equation (43) and (42) gives

$$G = \int_0^1 \pi^*(s^*)(1 - s^*)\,\mathrm{d}s^* \tag{50}$$

Since $\pi^*(s^*)$ has a constant sign within the limits of the integral it is clear that G takes the sign of $\pi(s_{n+1})$. The theorem 5 is therefore true.

Consider theorem 6

Equation (49) can only hold if the two integrals have the opposite sign. This is contrary to equation (38). Theorem 6 is therefore true.

Consider theorem 7

If the beam is built in at the left end ($s = 0$), equation (16) is for $i = k$

$$\int_0^S \pi(s)\,\mathrm{d}s = 0. \tag{51}$$

The theorems 1–4 insures (see also property 1 of the minimum weight beam) that there is at least $(k-1)$ zeros in the bending moment. If there is more than $(k-1)$ zeros there must be one interval with at least two zeros. In this case the theorem is clearly true. If the number of zero is equal to $(k-1)$ equation (38) must hold at $s = 0$.

$$\operatorname{sgn}\left(\int_0^S \pi(s)\,\mathrm{d}s\right) = \operatorname{sgn}(\pi(0)) \tag{52}$$

Hence from (51) above, it follows that

$$\pi(0) = 0.$$

It then follows from the above theorems that there must be either two zeros in the last interval (one at $s = 0$ and one at $s_0 < s < s_1$) or one in the last interval (at $s = 0$) and two in another interval. The theorem 7 is therefore true.

The properties of the minimum weight beam given in the main text will now be related to the above theorems. Consider properties 1, 2 and 3 first.

Starting at the right end of the beam ($s = S$) moving to the left, theorem 1 states that there must be at least one zero in the first interval. If there is just one zero in this interval theorem 2 and 3 state that there must be at least one zero in the next interval. If there is one zero in this interval there must be at least one in the next and so on. It follows from theorems 5 and 4 that an interval without any zeros can only occur if there is a previous interval with at least two zeros. Furthermore 5 and 2 state that an interval without zeros must be followed by one with at least one zero. The total number of zeros after n intervals must therefore be at least n.

If the beam is simply supported at the left end the number of intervals is equal to the degree of redundancy. In this case property 4 of the minimum weight beam follows directly from the above.

If the beam is built in at this end property 4 follows from this argument and theorem 7.

(*Received* 26 *October* 1970; *revised* 4 *February* 1971)

Абстракт—Если шарниры без трения расположенные во всех точках балкаи, где изгибающий момент равняется нулю, оказывается, что число и положение этих щарниров дают возможность формулировки расчета на минимум веса, в упругой или пластической областях способом статически определимым. Этот факт можно использовать для частных расчетов. При пластическом расчете механизм разрушения таков, что нет изменение в наклоне шарниров. В рамках упругого расчета деформированная балка, вообще, имеет изменение наклона в этих точках, но даже, при специальных условиях изменение равняется нулю. Таким образом расчет ц упругой как и в пластической стадиях вполне эффективный.